MAT126

Module 10

- Integration with Trigonometric Identities

see Δ's to
Example #6
write up

Trigonometric Identities - Review

Pythagorean Identities

$\sin^2 x + \cos^2 x = 1$ $\Rightarrow$ $\cos^2 x = 1 - \sin^2 x$

$\Rightarrow$ $\sin^2 x = 1 - \cos^2 x$

$\sec^2 x = 1 + \tan^2 x$ $\Rightarrow$ $\tan^2 x = \sec^2 x - 1$

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = 2\cos^2 x - 1$ $\xrightarrow{\text{solve for } \cos^2 x}$ $\cos 2x + 1 = 2\cos^2 x$

$\frac{1}{2}(\cos 2x + 1) = \cos^2 x$

$= 1 - 2\sin^2 x$

solve for $\sin^2 x$

$-\frac{1}{2}(\cos 2x - 1) = \cancel{-2}\sin^2 x$ $\Rightarrow$ $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

Example #1

ex. evaluate $\int \cos^3 x\, dx$ $= \int \cos^2 x \cos x\, dx$

$= \int (1-\sin^2 x) \cos x\, dx$

$= \int (1-u^2)\, du$

$= u - \dfrac{u^3}{3} + C$

$= \boxed{\sin x - \frac{1}{3}\sin^3 x + C}$

RECALL:

$\int \sin x \cos x\, dx$

$= \int u\, du$

$$\cos^2 x = 1 - \sin^2 x$$

$\boxed{\begin{array}{l} u = \sin x \\ du = \cos x\, dx \end{array}}$

$$= \boxed{\sin x - \frac{1}{3}\sin^3 x + C}$$

Example #2

ex. find $\int_0^{\pi} \sin^2 x \, dx$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$= \int_0^{\pi} \frac{1}{2}(1 - \cos 2x)\,dx$$

$$= \frac{1}{2}\int_0^{\pi}(1 - \cos 2x)\,dx$$

$$= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)\Big|_0^{\pi}$$

$$= \frac{1}{2}\left(\pi - 0 - \frac{1}{2}\left(\cancel{\sin 2\pi}^{\,0} - \cancel{\sin 0}^{\,0}\right)\right)$$

$$= \frac{1}{2}(\pi)$$

$$\int \cos(\overset{u}{2x})\,dx \qquad u = 2x \qquad \frac{1}{2}du = dx$$

$$= \frac{1}{2}\int \cos u \, du$$

$$= \frac{1}{2}\sin u \,(+C)$$

$$= \frac{1}{2}\sin(2x)$$

$$= \boxed{\frac{\pi}{2}}$$

Example #3

ex. find $\int_{\pi/2}^{3\pi/4} \sin^5 x \cos^3 x \, dx$

$$\frac{2^{8/2}}{2^8} = \frac{2^4}{2^8} = \frac{1}{2^4} = \frac{1}{16}$$

$$\boxed{\cos^2 x = 1 - \sin^2 x}$$

$$= \int_{\pi/2}^{3\pi/4} \sin^5 x \, \underbrace{\cos^2 x} \, \underbrace{\cos x \, dx}$$

$$= \int_{\pi/2}^{3\pi/4} \sin^5 x \left(1 - \sin^2 x\right) \cos x \, dx$$

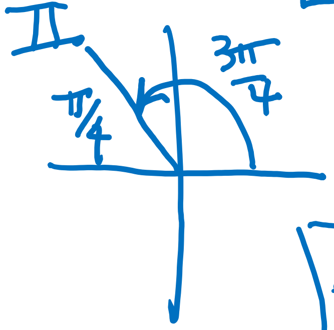
DISTRIBUTE

$$= \int_{\pi/2}^{3\pi/4} \left(\sin^5 x - \sin^7 x\right) \cos x \, dx$$

$u = \sin x$

$du = \cos x \, dx$

$u_{x=\frac{\pi}{2}} : \sin \frac{\pi}{2} = 1$



$u_{x=\frac{3\pi}{4}} : \sin \frac{3\pi}{4} = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$$= \int_{1}^{\sqrt{2}/2} \left(u^5 - u^7\right) du$$

$$= \frac{1}{6} u^6 \Big|_1^{\sqrt{2}/2} - \frac{1}{8} u^8 \Big|_1^{\sqrt{2}/2}$$

$$\boxed{\frac{\sqrt{2}^6}{2^6} = \frac{2^{6/2}}{2^6} = \frac{2^3}{2^6} = \frac{1}{2^3} = \frac{1}{8}}$$

$$= \frac{1}{6}\left(\left(\frac{\sqrt{2}}{2}\right)^6 - 1\right) - \frac{1}{8}\left(\left(\frac{\sqrt{2}}{2}\right)^8 - 1\right)$$

$$= \frac{1}{6}\left(\frac{1}{8} - \frac{8}{8}\right) - \frac{1}{8}\left(\frac{1}{16} - \frac{16}{16}\right)$$

$$= \frac{-1}{6}\left(\frac{7}{8}\right) + \frac{1}{8}\left(\frac{15}{16}\right)$$

$$\frac{-7}{2 \cdot 3 \cdot 8} \frac{(8)}{8} + \frac{15}{8 \cdot 8 \cdot 2} \frac{3}{3}$$

$$= \frac{-56 + 45}{384}$$

$$= \boxed{-\frac{11}{384}}$$

Example #4

$(\sin x\cos x)^2$

ex. find $\int_0^{\pi/2} \sin^2 x\cos^2 x\,dx$

$$= \frac{1}{4}\int_0^{\pi/2} 4\sin^2 x\cos^2 x\,dx$$

$$= \frac{1}{4}\int_0^{\pi/2} \sin^2 2x\,dx$$

$$= \frac{1}{4}\cdot\frac{1}{2}\int_0^{\pi} \sin^2 u\,du$$

$$= \frac{1}{8}\cdot\frac{1}{2}\int_0^{\pi}(1-\cos 2u)\,du$$

$$= \frac{1}{16}\left(u-\frac{1}{2}\sin 2u\right)\Big|_0^{\pi}$$

$$= \frac{1}{16}\left(\pi-0-\frac{1}{2}\left(\cancel{\sin 2\pi}^{\,0}-\cancel{\sin 0}^{\,0}\right)\right)$$

$$= \boxed{\frac{\pi}{16}}$$

$u = 2x$

$\frac{1}{2}du = dx$

$u_{x=0} = 0$

$u_{x=\frac{\pi}{2}} = \frac{2\cdot\pi}{2} = \pi$

$\frac{1}{4}\cdot 4 = 1$

$(\sin 2x)^2 = 4\sin^2 x\cos^2 x$

$\sin 2x = 2\sin x\cos x$

$\sin^2 u = \frac{1}{2}(1-\cos 2u)$

TRY:

$\int \sin^2 x\cos x\,\underbrace{\cos x\,dx}_{du}$

$$= \boxed{\frac{\pi}{16}}$$

Example #5

$\swarrow 4'd$

ex. evaluate $\int \tan^3 x \sec x \, dx$

$= \int \underbrace{\tan^2 x}\,\underbrace{\sec x \tan x \, dx}_{du}$

$= \int (\sec^2 x - 1)\, \underbrace{\sec x \tan x \, dx}$

$= \int (u^2 - 1)\, du$

$= \frac{u^3}{3} - u + C$

$= \boxed{\frac{1}{3} \cdot \sec^3 x - \sec x + C}$

$u = \sec x$

$du = \sec x \tan x \, dx$

$\tan^2 x = \sec^2 x - 1$

$\sec^2 x = 1 + \tan^2 x$

$(\tan x)' = \sec^2 x$

$\rightarrow (\sec x)' = \sec x \tan x$

$= \boxed{\frac{1}{3}\sec^3 x - \sec x + C}$

Example #6

ex. evaluate $\int \frac{dx}{x^2\sqrt{4-x^2}}$ $\xrightarrow[\text{ITO } \theta]{\text{write}}$ $\frac{2}{4}\int \frac{\cos\theta}{\sin^2\theta\sqrt{4-4\sin^2\theta}}\,d\theta$

$= \frac{1}{2}\int \frac{\cos\theta}{\sin^2\theta\sqrt{4(1-\sin^2\theta)}}\,d\theta$

$2\sqrt{1-\sin^2\theta}$

$= \frac{1}{2}\cdot\frac{1}{2}\int \frac{\cancel{\cos\theta}}{\sin^2\theta\,\boxed{\cancel{\cos^2\theta}}}\,d\theta$

$= \frac{1}{4}\int \frac{1}{\sin^2\theta}\,d\theta$

$= \frac{1}{4}\int \csc^2\theta\,d\theta$

$= -\frac{1}{4}\cot\theta + C$ $\Big\downarrow$ write ITO x

$= -\frac{1}{4}\frac{\cos\theta}{\sin\theta} + C$

$= -\frac{1}{4}\frac{\sqrt{4-x^2}/\cancel{2}}{x/\cancel{2}} + C = -\frac{\cancel{2}}{4}\frac{\sqrt{4-x^2}}{x} + C = \boxed{-\frac{\sqrt{4-x^2}}{4x} + C}$

$\frac{\cos\theta}{\sqrt{\cos^2\theta}} = \frac{\cancel{\cos\theta}}{\cancel{\cos\theta}} = 1$

$-(\cot x)' = +\csc^2 x$

$x = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$

$x^2 = 4\sin^2\theta$

$\frac{x}{2} = \sin\theta$

$\sqrt{4-x^2} = 2\cos\theta$